Oldroyd's Viscosity Result Extended to Circular Disk Particles Dispersed in Newtonian Fluids

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ABSTRACT: A viscosity equation is formulated on the basis of Oldroyd's theory for elastic and viscous properties of emulsions and suspensions by considering the drops as cylindrical rather than spherical in shape. The problem is formulated in three dimensions using cylindrical coordinates. The result can be considered as applicable to liquid, circular disk particles with negligible thickness, such as platelets, in dilute suspensions. In the present analysis, initially, stress effects are assumed uniform along the length of the cylinder, the z coordinate of velocity decays exponentially with time, and the interactive effects of the particles are assumed negligible. The independent parameters are viscosity of the solvent and dispersant, η and η', respectively, ratio of cylindrical radii a/b, a time derivative $\Delta = d/dt$, and constant interfacial tension γ. The present analysis is considered as the first step in analyzing the general problem of liquid cylinders suspended in liquids. Several different conclusions are reached compared with Oldroyd's model.

1. Introduction

Cylindrical fibers are used in plastic industries for composite materials, and this area has been the focus of much research, both experimental and theoretical. The early part of the development of this area is well documented by Frisch and Simha.[1] More recently, Ganani and Powell[2] have given an extensive review of experimental studies. Powell[3] has given an overview and summary of recent theoretical and experimental findings of present and past research groups in the field. Much of the work, both practical and analytical, is concerned with solid rods, such as glass fibers, having a finite aspect ratio. A basic problem is to find the interrelation between aspect ratio a_r (the ratio of major and minor lengths of the particle), volume fraction ϕ, and the alignment of the rods. Experimental techniques such as the falling ball rheometry of Milliken, Powell, Mondy and co-workers,[4-6] with randomly aligned rods have shown accurately the transition of viscosity from nonlinear to linear regions in which the fluid becomes semidilute. Specific and relative viscosity have also been measured with varying volume fraction for aligned rods. The hydrodynamic fact that aligned rods experience a lower viscosity than randomly oriented rods was confirmed in the experiments of Mondy et al.[6]

It can be assumed that as the aspect ratio decreases to one and lower the resultant effect will be that of a transition to disk-shaped particles, for example, circular disks. Simha[7] gave a classical analysis of the viscosity of solutions containing rigid oblate and prolate ellipsoids, obtaining the following viscosity relation (disks)

$$\eta^* = \eta(1 + \vartheta\phi) \qquad \vartheta = \frac{16}{15}\frac{f}{\tan^{-1} f} \qquad f = \frac{a_2}{a_1} \gg 1 \tag{1}$$

where f is the aspect ratio of the disk with thickness and radius a_1 and a_2, respectively. It can be observed (1), that for a fixed aspect ratio, the resultant viscosity η^* varies linearly with concentration ϕ. We have $\eta^* \rightarrow \eta$ for any solution valid as $a_1 \rightarrow 0$ or $\phi \rightarrow 0$. If we assume an expansion similar to (1) such that ϕ and ϑ are independent of a_1, as in the case of liquid cylinders of negligible thickness, then $\eta^* \rightarrow \eta$ as $a_2 \rightarrow 0$ or $\phi \rightarrow 0$. The independence of the solution on a_1 may be replaced with another condition, such as exponential decay (time $t \rightarrow \infty$) of the velocity component in the z (or a_1) direction. This component has coefficients that depend on the following positive parameters: time t, interfacial tension γ, cylindrical radius a, and the viscosities η and η' of solvent and dispersant, respectively. Thus, conditions are placed on a solution valid as $a_1 \rightarrow 0$, with respect to the viscosity η' of the liquid disks, since rigid disks correspond with $\eta' \rightarrow \infty$.

A parallel relation to (1) was also found by Simha[7] for rods

$$\vartheta = \frac{f^2}{15\left(\log 2f - \frac{3}{2}\right)} + \frac{f^2}{5\left(\log 2f - \frac{1}{2}\right)} + \frac{14}{15}$$

which along with other related expressions are documented in ref 1. Disklike objects can be viewed from a geometrical point of view as cylinders of small or negligible thickness h, which suggests a three-dimensional problem. In the limit as $h \rightarrow 0$ the problem of determining viscosity relations for suspensions of thin coin-shaped objects flowing in Newtonian or non-Newtonian liquids is essentially two-dimensional.

It is of significant interest to note Einstein's relation[8] is

$$\eta^* = \eta\left(1 + \frac{5}{2}\phi\right) \tag{2}$$

for small spherical particles dispersed in a Newtonian fluid of viscosity η and volume fraction ϕ. Taylor's formula[9] is

$$\eta^* = \eta\left[1 + \phi\left(\frac{\frac{5}{2}\eta' + \eta}{\eta' + \eta}\right)\right] \tag{3}$$

in which η' and η are the viscosities of the dispersant and the solvent, respectively. Experimental results by Taylor[10] verified (3) for slow flows using viscous materials such as nonsoluble droplets of oils, tar, or pitch in a dilute sugar solution.

Oldroyd[11] derived the relative viscosity relation for a suspension of liquid spheres of radius a, having viscosity η', in a solvent contained within a sphere of radius b, having viscosity η

$$\frac{\eta^*}{\eta} = \{40(\eta + \eta')\gamma + a(3\eta + 2\eta')(16\eta + 19\eta')\Delta + 3c[4(2\eta + 5\eta')\gamma - a(\eta - \eta')(16\eta + 19\eta')\Delta]\}/\{40(\eta + \eta')\gamma + a(3\eta + 2\eta')(16\eta + 19\eta')\Delta - 2c[4(2\eta + 5\eta')\gamma - a(\eta - \eta')(16\eta + 19\eta')\Delta]\} \tag{4}$$

The volume fraction $c = a^3/b^3$, where a/b is the ratio of sphere radii and η^* is the resultant viscosity of the

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emulsion. Oldroyd's result (4) has been employed in the experimental work of Graebling and Muller[12] that examined viscosity characteristics of PDMS/POE-DO polymers (poly(dimethylsiloxane)/poly(oxyethylenediol)). The experimental data compared favorably with the analytical results of the storage and loss moduli (see pp 116 of ref 13) G' and G'' calculated from (4). In the derivation of (4) Oldroyd derived an operator of the form

$$\eta^* = \eta_0 \frac{1 + \lambda_2\Delta + \delta_2\Delta^2 + \ldots}{1 + \lambda_1\Delta + \delta_1\Delta^2 + \ldots} \qquad \left(\Delta = \frac{d}{dt}\right)$$

where η_0, λ_i, and δ_i are material constants to be determined from the properties of the respective Newtonian fluids and the Newtonian stress equation: $\tau = -PI + 2\eta^* E$, such that P is the pressure, η^* is the constant suspension viscosity, I is the identity matrix, and E is the strain-rate tensor. The above operator employed in the stress equation accounts for the linear viscoelastic behavior of the suspension.

In the present paper we derive equations similar to (3) and (4) but using cylindrical coordinates (r, θ, z) and the Navier–Stokes equations for cylindrical coordinates. Initially, we consider the problem of fluid flow for liquid, cylindrical fiber suspensions, assuming that the effects of tangential and normal stresses are uniform along the length of the fiber and the motion of the fiber is in the z direction. Steady-state boundary conditions are applied to the w velocity component, and we find we place conditions on the viscosity η' so that the velocity is uniform in the z direction and the normal stress τ_{zz} is independent of the z-coordinate.

In section 2 we write the governing equations for a viscous, incompressible fluid in cylindrical coordinates. We give a general solution for this system of equations in section 3. In section 4 the boundary conditions of the two-dimensional domain are formulated. A curvature relation is found which expresses the change in curvature along the cylinder walls due to normal stress. In section 5, we calculate an equation similar to (4), assuming the existence of liquid, circular disk particles of viscosity η', having radius a and negligible thickness h, suspended in a Newtonian fluid of viscosity η which is contained within a circular particle of radius b. The resultant viscosity η^* of the suspension is calculated as the perturbing effects the emulsion would have on a larger circular disk particle of radius R. In section 6 we give a comparison of the new result with Oldroyd's equation (4) and calculate relations for the relaxation parameters λ_1 and λ_2.

2. Governing Equations

The system of equations governing the plane, slow flow of a viscous, incompressible fluid in cylindrical coordinates is given by (see pp 1 and 25 of ref 14)

$$\frac{\partial(ru)}{\partial r} + \frac{\partial \nu}{\partial \theta} + r\frac{\partial w}{\partial z} = 0 \qquad (5)$$

$$\frac{\partial P}{\partial r} = \eta^*\left[\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} - \frac{u}{r^2} - \frac{2}{r^2}\frac{\partial \nu}{\partial \theta} + \frac{\partial^2 u}{\partial z^2}\right] \qquad (6)$$

$$\frac{1}{r}\frac{\partial P}{\partial \theta} = \eta^*\left[\frac{\partial^2 \nu}{\partial r^2} + \frac{1}{r}\frac{\partial \nu}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \nu}{\partial \theta^2} - \frac{\nu}{r^2} + \frac{2}{r^2}\frac{\partial u}{\partial \theta} + \frac{\partial^2 \nu}{\partial z^2}\right] \qquad (7)$$

$$\frac{\partial P}{\partial z} = \eta^*\left[\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2} + \frac{\partial^2 w}{\partial z^2}\right] \qquad (8)$$

where u, ν, w, and P are respectively the velocity components and pressure. η^* denotes viscosity of a resulting suspension in (6)–(8). The components of stress are

$$\tau_{rr} = -P + 2\eta^*\frac{\partial u}{\partial r} \qquad \tau_{\theta\theta} = -P + 2\eta^*\left(\frac{1}{r}\frac{\partial \nu}{\partial \theta} + \frac{u}{r}\right)$$
$$\tau_{r\theta} = \eta^*\left(\frac{1}{r}\frac{\partial u}{\partial \theta} + \frac{\partial \nu}{\partial r} - \frac{\nu}{r}\right) \qquad (9)$$

$$\tau_{zz} = -P + 2\eta^*\frac{\partial w}{\partial z} \qquad \tau_{\theta z} = \eta^*\left(\frac{1}{r}\frac{\partial w}{\partial \theta} + \frac{\partial \nu}{\partial z}\right)$$
$$\tau_{rz} = \eta^*\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial r}\right) \qquad (10)$$

The general form of the stress tensor is a symmetric 3 × 3 matrix with six unknowns and components given in (9) and (10). The assumption of linearity of the solution at the boundary $r = R$ and the principle of superposition applied to a change of basis imply that it is sufficient to consider a diagonal stress matrix of the form

$$(\tau)_{a,b} = \mathrm{diag}\{T_0 - T_1, -T_0 - T_1, 2T_1\}$$

such that T_0 and T_1 are constant and

$$\tau_{rr} = T_0 \cos(2\theta) - T_1 \qquad \tau_{\theta\theta} = -T_0 \cos(2\theta) - T_1$$
$$\tau_{zz} = 2T_1 \qquad \tau_{r\theta} = -T_0 \sin(2\theta)$$

$$u = \frac{r}{2\eta^*}(T_0 \cos(2\theta) - T_1) \qquad \nu = -\frac{r}{2\eta^*}T_0 \sin(2\theta)$$
$$w = \frac{T_1}{\eta^*}z \qquad P = 0$$

In the next section we write the general solution to the system (5)–(8) in terms of six coefficients.

3. General Solution

The assumption on the flow is that after steady state has been reached, the coefficients in the solution of the system (5)–(8) depend on time because of the introduction of relaxation and retardation parameters λ_1 and λ_2. We also assume a linear function of z for the w component of velocity in order to maintain the assumption of a uniform stress along the axis of the cylinder. The general solution to the system (5)–(8), in terms of the velocity components u, ν, w, and pressure P is (deleting *)

$$u = [Ar + Br^{-3} + Cr^3 + Dr^{-1}] \cos(2\theta) - A_1 r/2 \qquad (11)$$

$$\nu = -[Ar - Br^{-3} + 2Cr^3] \sin(2\theta) \qquad (12)$$

$$w = A_1 z + A_0 \qquad (13)$$

$$P = 2\eta[3Cr^2 + Dr^{-2}] \cos(2\theta) + P_0 \qquad (14)$$

where the coefficients A_0, A_1, A, B, C, D, and P_0 depend on time t. The normal and tangential stress components τ_{rr}, $\tau_{r\theta}$, $\tau_{\theta\theta}$, and τ_{zz} found by employing (9)–(10) in (11)–(14) are given by

$$\tau_{rr} = -P_0 + 2\eta[A - 3Br^{-4} + 3Cr^2 - Dr^{-2}] \cos(2\theta) - \eta A_1 \qquad (15)$$

$$\tau_{r\theta} = -2\eta[A + 3Br^{-4} + 3Cr^2 + Dr^{-2}] \sin(2\theta) \qquad (16)$$

$$\tau_{\theta\theta} = -P_0 + 2\eta[-A + 3Br^{-4} - 3Cr^2 + Dr^{-2}] \cos(2\theta) - \eta A_1 \qquad (17)$$

$$\tau_{zz} = -P_0 + 2\eta A_1 \qquad (18)$$

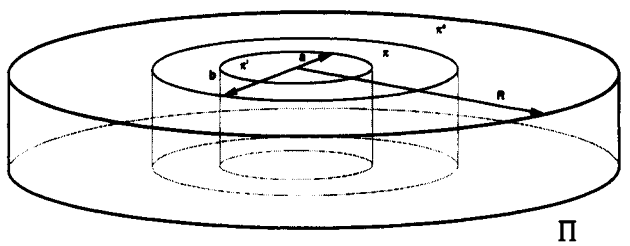

Figure 1. Circular disk particle.

The relative sizes of the coefficients A–D are assumed to be $C = O(a^4/b^6A)$, $B = O(a^4A)$, $D = O(a^2A)$. This ensures that the approximate expansion of u, ν, and P in powers of the parameter fraction $c = a^2/b^2$ is valid. In the next section we calculate the curvature of the cylinder surface and then determine the boundary conditions.

4. Boundary Conditions

We consider the three-dimensional region Π consisting of subregions π', π, and π^* in Figure 1, with varying radii a, b, and R, respectively, and thickness h such that $0 < a < b \ll R < \infty$. The unknown functions in Π are comprised of $A, A_i, B, C, D, A', A_i', B', C', D', A^*, A_i^*, B^*, C^*, D^*, P_0, P_0', P_0^*$, and η^*, $i = 0, 1$.

The velocity field is everywhere finite ($r = 0$) which implies

$$B' = D' = 0 \tag{19}$$

In order tht $B \neq 0$ and $D \neq 0$ we require the condition $A_1 \to 0$ as $t \to \infty$ which removes the singularity at $r = 0$. To maintain a velocity approximately linear in r as $r \to R$ we have

$$C^* = D^* = 0 \tag{20}$$

The displacement of the cylinder ($r = a$) is assumed to take the form of the velocity components for large values of the interfacial tension γ, so that the velocity components are given by

$$u = \frac{dr}{dt} \qquad \nu = a\frac{d\theta}{dt} \qquad w = \frac{dz}{dt} \tag{21}$$

Inspection of (11)–(13) using (21) yields

$$r = a\left(1 - \frac{A_1''}{2}\right) + [A''a + C''a^3] \cos (2\alpha) \tag{22}$$

$$\theta = \alpha - [A'' + 2C''a^2] \sin (2\alpha) \tag{23}$$

$$z = A_1''h + A_0'' \tag{24}$$

where $\Delta A_i' = A_i'$, $\Delta C'' = C'$, and $\Delta A'' = A'$, $i = 0, 1$. Without loss of generality we take $A_0'' = 0$.

In order to calculate the discontinuity in the stress components we introduce a two-parameter cylindrical surface

$$X(\alpha,h) = \langle r(\alpha) \cos \theta(\alpha), r(\alpha) \sin \theta(\alpha), z(h) \rangle$$

where $r = r(\alpha)$ and $\theta = \theta(\alpha)$ are arbitrary functions of α and $z = z(h)$ is a function of h. The principal curvature κ, of the surface, is found to be

$$\kappa = \frac{-[2(r')^2\theta' + rr'\theta'' - rr''\theta' + r^2(\theta')^3]}{[(r')^2 + r^2(\theta')^2]^{3/2}} \tag{25}$$

where prime (′) denotes differentiation with respect to α. Substituting (22)–(24) into (25) and disregarding the quadratic terms in the products of A'', C'', and A_1'' we obtain

$$\kappa \approx \frac{1}{a}\left\{1 + \frac{A_1''}{2} + 3[A'' + C''a^2] \cos (2\alpha)\right\} \tag{26}$$

It is well-known (see p 12 of ref 15) that the discontinuity in the pressure P_0 is inversely proportional to the radius of curvature $|\kappa|$ such that the interfacial tension is the constant of proportionality. Using this and (26) in (15) yields two equations

$$P_0 = P_0' - \frac{\gamma}{a} \tag{27}$$

$$-\frac{\gamma}{a}A_1'' = 2(\eta A_1 - \eta' A_1') \tag{28}$$

Applying the operator Δ, we find the continuity of τ_{rr} across $r = a$ is expressed by

$$\eta\Delta[A - 3Ba^{-4} + 3Ca^{-2} - Da^{-2}] = \left[a^{-1}\gamma\frac{3}{2} + \eta'\Delta\right]A' + \left[a\gamma\frac{3}{2} + \eta' 3a^2\Delta\right]C' \tag{29}$$

Similarly, the conditions for $\tau_{r\theta}$, u, and ν are given by

$$\eta[A + 3Ba^{-4} + 3Ca^2 + Da^{-2}] = \eta'[A' + C'a^2] \tag{30}$$

$$A_1 = A_1' \tag{31}$$

$$Aa + Ba^{-3} + Ca^3 + Da^{-1} = Aa' + C'a^3 \tag{32}$$

$$Aa - Ba^{-3} + 2Ca^3 = A'a + 2C'a^3 \tag{33}$$

Applying Δ to (28) and employing (31), we obtain a first-order linear differential equation in t with solution

$$A_1 = k \exp\left[\left(\frac{-\gamma}{2a(\eta - \eta')}\right)t\right] \qquad k = \text{constant} \tag{34}$$

The condition that $A_1 \to 0$ as $t \to \infty$ if $\eta > \eta'$ follows from (34). Across the interface $r = b$, from (15), we find

$$\eta A_1 = \eta^* A_1^* \tag{35}$$

Other continuity conditions are

$$\eta[A - 3Bb^{-4} + 3Cb^2 - Db^{-2}] = \eta^*[A^* - 3B^*b^{-4}] \tag{36}$$

$$\eta[A + 3Bb^{-4} + 3Cb^2 + Db^{-2}] = \eta^*[A^* + 3B^*b^{-4}] \tag{37}$$

$$A_1^* = A_1 \tag{38}$$

$$Ab + Bb^{-3} + Cb^3 + Db^{-1} = A^*b + B^*b^{-3} \tag{39}$$

$$Ab - Bb^{-3} + 2Cb^3 = A^*b - B^*b^{-3} \tag{40}$$

$$P = P^* \tag{41}$$

It follows that $\eta = \eta^*$ from (35) and (38). However, this equality may be neglected if A_1 is sufficiently small or, equivalently by (34), t is sufficiently large. Additional boundary conditions are a zero pressure at $r = R$

$$P^* = 0 \tag{42}$$

$$A^* = \frac{T_0}{2\eta^*} \tag{43}$$

The required boundary conditions are expressed through (19)–(20) and (27)–(43). The velocity at the boundary $r = R$ is found by substituting (43) into (11) and (12) to yield

$$u = \frac{T_0}{2\eta^*} r\left(1 + \frac{2B^*\eta^*}{T_0 r^4}\right)\cos(2\theta)$$

$$v = -\frac{T_0}{2\eta^*} r\left(1 - \frac{2B^*\eta^*}{T_0 r^4}\right)\sin(2\theta) \quad (44)$$

The corresponding stress components, at the boundary $r = R$, are given by

$$\tau_{rr} = T_0\left(1 - \frac{6\eta^* B^*}{T_0 r^4}\right)\cos(2\theta)$$

$$\tau_{r\theta} = -T_0\left(1 + \frac{6\eta^* B^*}{T_0 r^4}\right)\sin(2\theta) \quad (45)$$

Inspection of (44) and (45) implies that the error in the velocity and stress components is $O(R^{-4})$ which is presumed to be acceptable. Also, it is assumed that steady state is reached ($A_1 \to 0$) so that the problem is essentially two-dimensional, independent of the z coordinate. In the next section, using a similar algebraic technique, first employed by Oldroyd,[11] we calculate an equation for η^*.

5. Calculation of the Viscosity Ratio η^*/η

An operator η^*/η is found, in terms of parameters η', η, γ, a, b, and $\Delta = d/dt$ by algebraic methods. Adding and subtracting (32) and (33) gives

$$\frac{D}{a^2} = 2(A' - A) + 3(C' - C)a^2 \quad (46)$$

$$\frac{B}{a^4} = -(A' - A) - 2(C' - C)a^2 \quad (47)$$

Substituting (46) and (47) into (29) and (30) gives

$$\left[a^{-1}\gamma\frac{3}{2} + (\eta' - \eta)\Delta\right]A' + \left[a\gamma\frac{3}{2} + 3(\eta' - \eta)a^2\Delta\right]C' = 0 \quad (48)$$

$$2\eta[A + 3Ca^2] = (\eta' + \eta)[A' + 3C'a^2] = 0 \quad (49)$$

Solving (48) and (49) for A' and C' in terms of A and C we find, taking $\Gamma = (\eta' + \eta)a^{-1}\gamma$

$$\Gamma A' = -\eta[a^{-1}\gamma + 2(\eta' - \eta)\Delta][A + 3Ca^2] \quad (50)$$

$$a^2\Gamma C' = \frac{\eta}{3}[a^{-1}\gamma 3 + 2(\eta' - \eta)\Delta][A + 3Ca^2] \quad (51)$$

Eliminating the primed variables A' and C' from (48) and (49) by use of (50) and (51) yields

$$\frac{\Gamma D}{a^2} = -[a^{-1}\gamma(\eta + 2\eta') + 2\eta(\eta' - \eta)\Delta]A - [a^{-1}\gamma 3\eta' + 6\eta(\eta' - \eta)\Delta]Ca^2 \quad (52)$$

$$\frac{\Gamma B}{a^4} = \left[a^{-1}\gamma\eta' + \frac{2\eta}{3}(\eta' - \eta)\Delta\right]A + [a^{-1}\gamma(2\eta' - \eta) + 2\eta(\eta' - \eta)\Delta]Ca^2 \quad (53)$$

Adding and subtracting (39) and (40) we obtain

$$B^* b^{-3} = Bb^{-3} - \frac{1}{2}Cb^3 + \frac{1}{2}Db^{-1} \quad (54)$$

$$A^* b = Ab + \frac{3}{2}Cb^3 + \frac{D}{2}b^{-1} \quad (55)$$

Employing (54) and (55) in (36) and (37) we find

$$\eta^*[A + 3Bb^{-4} + 2Db^{-2}] = \eta[A + 3Bb^{-4} + 3Cb^2 + Db^{-2}] \quad (56)$$

$$\eta^* = \eta \quad (57)$$

The solution $\eta^* = \eta$ is assumed valid for $a \to 0$ or $h \to 0$. To determine the equation for viscosity η^*/η from (56), we see that $C = O[(a^4/b^6)A]$, $c = a^2/b^2$, such that terms of order c^2 in the subsequent analysis are neglected, so that, applying the operator Γ to (56), we find

$$\frac{\eta^*}{\eta} = \frac{(\eta' + \eta)\gamma - c[\gamma(\eta + 2\eta') + 2a\eta(\eta' - \eta)\Delta]}{(\eta' + \eta)\gamma - 2c[\gamma(\eta + 2\eta') + 2a\eta(\eta' - \eta)\Delta]} \quad (58)$$

The approximation (58) is analogous to Oldroyd's result (4). In the next section we discuss the new result (58).

6. Discussion

Inspection shows (58) has some similarities to (4): as in Oldroyd's model we find that the viscosity η^* is inversely proportional to the interfacial tension γ and directly proportional to the volume fraction c. However, the ratio η^*/η in (58) represents a different result than (4). Moreover, as with (4), (58) can be expressed as

$$\eta^* = \eta_0\frac{1 + \lambda_2\Delta}{1 + \lambda_1\Delta}$$

where η_0 represents a Newtonian viscosity factor which can be shown to be related to η by

$$\frac{\eta_0}{\eta} = 1 + 2\frac{\left(\frac{\eta}{2} + \eta'\right)}{\eta + \eta'}c + O(c^2) \quad (59)$$

with

$$\lambda_2 = \frac{2ac\eta}{\gamma}\frac{(\eta - \eta')}{\eta + \eta'} + O(c^2) \qquad \lambda_1 = \frac{4ac\eta}{\gamma}\frac{(\eta - \eta')}{\eta + \eta'} + O(c^2) \quad (60)$$

It follows $\lambda_1 > \lambda_2 > 0$ if and only if $\eta' < \eta$, which is consistent with the previous analysis. We find 2.5 in (3) is replaced by 2 in (59) which can be explained by the fact that the surface area of the sphere is $4\pi a^2$ and the surface area of the circular disk for sufficiently small h satisfies $2\pi a^2 + 2\pi ah \cong 2\pi a^2$.

It can be assumed that $0 < h < c^2$. We see that (59) is of the form

$$\eta^* = \eta(1 + \vartheta\phi)$$

such that the percentage volume concentration $c = \phi = (a/b)^2$ and factor ϑ are independent of disk thickness h: a_1 in (1). We see that there is no corresponding formula for rigid disks that can be derived from (59) since necessarily $\eta' < \eta$; however, we find (59), to first order, derives from (58) if $\gamma \to \infty$, similar to Oldroyd's model (4) with respect to (3). It is of interest to note that $\vartheta = 2.016$ for oblate spheroids is the minimum factor found in Jeffrey's classical paper.[16] This suggests a limiting value of approximately 2 for rigid disks which would be correct if the conditions on η' were removed; however, another detailed analysis such as discussed below is required to verify this.

An application of the preceding analysis is composite materials consisting of liquid, circular disk fibers dispersed in a Newtonian solvent such that the fibers have viscosity η' which is less than the viscosity η of the solvent. A further study of the present paper involves determining the viscosity for three-dimensional, liquid cylinders.

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